# Interaction of a moving charged particle with a spatially dispersive medium. I. Structure of the electromagnetic field

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# Interaction of a moving charged particle with a spatially dispersive medium. I. Structure of the electromagnetic field*

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The general structure of the electromagnetic field is determined which is generated by a uniformly moving point charge interacting with a spatially dispersive medium forming a plane parallel slab. The direction of the point charge is taken to be at right angles to the faces of the slab, and the dielectric constant of the medium in the wave-vector frequency space is assumed to be of the form $\epsilon_{ij}(\vec{k},\omega) = \delta_{ij}\{\epsilon_0(\omega) + \chi/[\vec{k}^2 - \mu^2(\omega)]\}$. Expressions for Čerenkov and transition radiation fields associated with a uniformly moving point charge passing from a vacuum into such a medium, forming a half space, are also obtained.

## I. INTRODUCTION

Although a uniformly moving charged particle in a vacuum does not radiate, it gives rise to a radiation in a material medium whenever its velocity exceeds the phase velocity of light in the medium. This radiation is the well-known Čerenkov radiation.[1] A different type of electromagnetic field is generated when the charged particle encounters a sharp boundary that separates different material media. Unlike the Čerenkov radiation, this radiation, though weak, occurs at all velocities of the charged particle. It is known as transition radiation.[2] Both Čerenkov and transition radiation can be described in the framework of macroscopic Maxwell's theory. Because of the important features of these radiations, several theoretical and experimental investigations have been made to study them in different material media. For a review and an extensive bibliography the reader is referred to an article by Bass and Yakovenko.[3]

In this paper we will consider the problem of Čerenkov and transition radiation associated with a uniformly moving charged particle passing from a vacuum into a model spatially dispersive half-space, whose dielectric function in wave-vector frequency space has the form

$$\epsilon(\vec{k},\omega) = \epsilon_0(\omega) + \frac{\chi}{\vec{k}^2 - \mu^2(\omega)} . \tag{1.1}$$

A dielectric function of the form (1.1) is appropriate for a dielectric near an exciton transition frequency.[4] For a detailed account of the dielectric function in a medium where excitons are produced by light we refer the reader to Ref. 5. In the case of metals, one also obtains a dielectric function of the form (1.1) when one is in the frequency regime where the hydrodynamic approximation is applicable to the dynamics of the electrons (see a paper by Heinrichs[6]). For a dielectric medium where spatial dispersion is due to an isolated exciton transition, $\chi$ and $\mu^2(\omega)$ are given by[7]

$$\chi = \frac{4\pi\alpha_e m_e^* \omega_e}{\hbar} ,$$
$$\mu^2(\omega) = \frac{m_e^*}{\hbar\omega_e}(\omega^2 - \omega_e^2 + i\omega\Gamma) . \tag{1.2}$$

Here $\alpha_e$ is the oscillator strength associated with the exciton transition, $m_e^*$ is the effective mass of the exciton, $\Gamma$ is the damping constant, and $\hbar$ is Planck's constant divided by $2\pi$. $\epsilon_0(\omega)$ is the background dielectric constant due to all other transitions and is assumed to be independent of the wave vector.

The electromagnetic field in spatially dispersive media whose dielectric function is of the form (1.1) has recently been studied by several authors.[8] In treating the problem of Čerenkov and transition radiation in spatially dispersive media we follow the work of Agarwal, Pattanayak, and Wolf.[8d]

In Sec. II we formulate the problem and write down the basis equations. In Sec. III we obtain the structure of the electromagnetic fields associated with a uniformly moving charged particle in the spatially dispersive model medium which has the form of a plane parallel slab. In Sec. IV we use the mode expansion derived in Sec. III to obtain expressions for the Čerenkov and transition radiation fields associated with a uniformly moving charged particle that is incident from vacuum onto the spatially dispersive model half space, in the direction normal to the interface.

In paper II of this investigation we present discussions of the Čerenkov field which we express in terms of modified Bessel functions of the third kind. We then obtain an asymptotic expansion and threshold conditions of the Čerenkov radiation. We also obtain asymptotic expansions for the transition radiation outside the spatially dispersive medium.

## II. BASIC FORMULATION

In this section we will write down the constitutive relation for our model spatially dispersive medium in $(\vec{r}, \omega)$ space. We will then obtain equations for electromagnetic field vectors in the presence of an external charge density $\rho(\vec{r}, t)$ and current density $\vec{J}(\vec{r}, t)$.

Let us first introduce the Fourier temporal transform of the electric field vector $\vec{E}(\vec{r}, t)$, the dielectric induction vector $\vec{D}(\vec{r}, t)$, the magnetic field vector $\vec{H}(\vec{r}, t)$, and others:

$$\vec{E}(\vec{r}, \omega) = \int_{-\infty}^{+\infty} \vec{E}(\vec{r}, t) e^{i\omega t} dt, \text{ etc.} \tag{2.1}$$

Similarly the spatial Fourier transform is introduced as

$$\vec{E}(\vec{k}, \omega) = \int\int\int_{-\infty}^{+\infty} \vec{E}(\vec{r}, t) e^{-i\vec{k}\cdot\vec{r}} d^3r, \text{ etc.} \tag{2.2}$$

Using (1.1) the relation between $\vec{D}(\vec{k}, \omega)$ and $\vec{E}(\vec{k}, \omega)$ for the spatially dispersive medium under consideration can be written as

$$\vec{D}(\vec{k}, \omega) = \left(\epsilon_0(\omega) + \frac{\chi}{\vec{k}^2 - \mu^2(\omega)}\right)\vec{E}(\vec{k}, \omega). \tag{2.3}$$

On taking the spatial Fourier transform of (2.3), we obtain

$$\vec{D}(\vec{r}, \omega) = \epsilon_0(\omega)\vec{E}(\vec{r}, \omega) + \frac{\chi}{4\pi}\int\int\int_{-\infty}^{+\infty} G_\mu(|\vec{r} - \vec{r}'|)\vec{E}(\vec{r}', \omega) d^3r', \tag{2.4}$$

where

$$G_\mu(|\vec{r} - \vec{r}'|) = \frac{e^{i\mu|\vec{r}-\vec{r}'|}}{|\vec{r} - \vec{r}'|}, \tag{2.5a}$$

where $\mu$ is the square root of the second expression in (1.2) for which

$$\operatorname{Im}\mu > 0. \tag{2.5b}$$

We further assume that the medium is nonmagnetic so that the magnetic induction vector $\vec{B}$ and the magnetic field vector $\vec{H}$ are equal to each other in the Gaussian system of units used throughout this paper, i.e.,

$$\vec{B}(\vec{r}, \omega) = \vec{H}(\vec{r}, \omega). \tag{2.6}$$

The constitutive relation (2.4) is applicable to an infinite medium. For a finite medium, the constitutive relation will differ in form from that of (2.4) near the boundary of the medium. However, as we are interested in bulk properties, and, as a matter of fact, we will later assume the charged particle to be moving uniformly throughout the whole space, the effect of the deviations of the form of the constitutive relation due to the presence of the boundaries will be assumed to be small. We will therefore take as our model medium a spatially dispersive medium whose constitutive relation in the spatial domain is given by the relation

$$\vec{D}(\vec{r}, \omega) = \epsilon_0(\omega)\vec{E}(\vec{r}, \omega) + \frac{\chi}{4\pi}\int_{\mathfrak{D}} G_\mu(|\vec{r} - \vec{r}'|)\vec{E}(\vec{r}', \omega) d^3r', \tag{2.7}$$

where $\mathfrak{D}$ denotes the domain occupied by the spatially dispersive medium, and the dimensions of which are assumed to be large compared to the range of the kernel $G_\mu(|\vec{r} - \vec{r}'|)$, i.e., $(\operatorname{Im}\mu)^{-1}$.

The macroscopic Maxwell's equations in the presence of an external charge density $\rho(\vec{r}, t)$ and current density $\vec{J}(\vec{r}, t)$ in the medium under consideration can be written in frequency space $\omega$ as follows:

$$\nabla\times\vec{E}(\vec{r}, \omega) - i\frac{\omega}{c}\vec{H}(\vec{r}, \omega) = 0 \tag{2.8a}$$

$$\nabla\times\vec{H}(\vec{r}, \omega) + \frac{i\omega}{c}\epsilon_0(\omega)\vec{E}(\vec{r}, \omega) = \frac{4\pi}{c}\vec{J}(\vec{r}, \omega) - \frac{i\omega\chi}{4\pi c}\int_{\mathfrak{D}} G_\mu(|\vec{r} - \vec{r}'|)\vec{E}(\vec{r}', \omega) d^3r' \tag{2.8b}$$

$$\nabla\cdot\vec{D}(\vec{r}, \omega) = 4\pi\rho(\vec{r}, \omega), \tag{2.8c}$$

$$\nabla\cdot\vec{B}(\vec{r}, \omega) = \nabla\cdot\vec{H}(\vec{r}, \omega) = 0, \tag{2.8d}$$

where

$$\vec{J}(\vec{r}, \omega) = \int_{-\infty}^{+\infty} \vec{J}(\vec{r}, t) e^{i\omega t} dt,$$

and

$$\rho(\vec{r}, \omega) = \int_{-\infty}^{+\infty} \rho(\vec{r}, t) e^{i\omega t} dt.$$

Equations (2.8a)-(2.8b) are a set of coupled, inhomogeneous integrodifferential equations in the Cartesian components of $\vec{E}$ and $\vec{H}$. It is to be noted that (2.8c) and (2.8d) follow from (2.8b) and (2.8a), respectively, for nonzero frequencies on taking the divergence of both sides of Eqs. (2.8a) and (2.8b) and making use of the law of conservation of charge, i.e.,

$$\nabla\cdot\vec{J}(\vec{r},\omega)=\frac{i\omega}{c}\rho(\vec{r},\omega). \tag{2.9}$$

We also note that once the electric field is determined, the expressions for the magnetic field can be obtained by Eq. (2.8a), viz.,

$$\vec{H}(\vec{r},\omega)=-\frac{ic}{\omega}\nabla\times\vec{E}(\vec{r},\omega). \tag{2.10}$$

In order to obtain an equation for $\vec{E}(\vec{r},\omega)$ alone we substitute (2.10) in (2.8b). We then find that

$$\nabla\times[\nabla\times\vec{E}(\vec{r},\omega)]-k_0^2\epsilon_0(\omega)\vec{E}(\vec{r},\omega)=\frac{4\pi i k_0}{c}\vec{J}(\vec{r},\omega)+\frac{k_0^2\chi}{4\pi}\int_{\mathfrak{D}}G_\mu(|\vec{r}-\vec{r}'|)\vec{E}(\vec{r}',\omega)d^3r', \tag{2.11}$$

where

$$k_0=\omega/c. \tag{2.12}$$

Equation (2.11) is the basic equation of our theory, which the macroscopic electric field generated by the current density must obey at all points inside the domain $\mathfrak{D}$.

## III. STRUCTURE OF THE ELECTROMAGNETIC FIELD GENERATED BY A UNIFORMLY MOVING CHARGED PARTICLE IN A PLANE PARALLEL SPATIALLY DISPERSIVE SLAB

In Eq. (2.11) the domain $\mathfrak{D}$ occupied by the spatially dispersive medium is of arbitrary geometry. The current which appeared in Eq. (2.11) can be due to an arbitrary charge distribution. In this section we will restrict our discussion to the case when the domain $\mathfrak{D}$ is a plane parallel slab of thickness $d$, and the current distribution is that due to a uniformly moving point charge in a direction perpendicular to the front face of the slab which is taken to be the $(x, y)$ plane. The $\hat{z}$ direction is taken to be the path of the charged particle. The current density associated with a point charge uniformly moving in the positive $\hat{z}$ direction can be represented by

$$\vec{J}(\vec{r},t)=ev\hat{z}\delta(z-vt)\delta(x)\delta(y), \tag{3.1}$$

where $e$ is the charge, $v$ is the velocity of the particle, and $\delta$ denotes the Dirac $\delta$ function. On taking the Fourier temporal transform of (3.1) we obtain

$$\vec{J}(\vec{r},\omega)=e\hat{z}e^{i(\omega/v)z}\delta(x)\delta(y). \tag{3.2}$$

On using (3.2) and taking the domain $D$ to be the space occupied by the slab, i.e., $-\infty<x<+\infty$, $-\infty<y<+\infty$, and $0<z<d$, Eq. (2.11) becomes

$$k_0^2\epsilon_0(\omega)\vec{E}(\vec{r},\omega)-\nabla\times[\nabla\times\vec{E}(\vec{r},\omega)]=-\frac{4\pi e i k_0}{c}\hat{z}e^{i(\omega/v)z}\delta(x)\delta(y)$$
$$-\frac{\chi k_0^2}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx'dy'\int_0^d dz' G_\mu(|\vec{r}-\vec{r}'|)\vec{E}(\vec{r}',\omega). \tag{3.3}$$

Since the slab extends from $-\infty$ to $+\infty$ in the $x$ and $y$ directions, we represent the electric field $\vec{E}(\vec{r},\omega)$ in the form of a two-dimensional Fourier transform in the $x$ and $y$ coordinates, i.e., as

$$\vec{E}(\vec{r},\omega)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\vec{E}(p,q;z,\omega)e^{i(px+qy)}dp\,dq. \tag{3.4}$$

On taking the two-dimensional Fourier transform in the $x$ and $y$ coordinates of both sides of Eq. (3.3) and on using (3.4), we obtain

$$k_0^2\epsilon_0(\omega)\vec{E}(p,q;z,\omega)-\left(ip,iq,\frac{\partial}{\partial z}\right)\times\left[\left(ip,iq,\frac{\partial}{\partial z}\right)\times\vec{E}(p,q;z,\omega)\right]$$
$$=-\frac{eik_0}{\pi c}\hat{z}\,e^{i(\omega/v)z}-\left(\frac{i\chi\omega^2}{2c^2}\right)\int_0^d\frac{e^{iw_\mu|z-z'|}}{w_\mu}\vec{E}(p,q;z',\omega)dz', \tag{3.5a}$$

where

$$w_\mu=(\mu^2-p^2-q^2)^{1/2}\text{ with }\mathrm{Re}w_\mu>0\text{ and }\mathrm{Im}w_\mu>0. \tag{3.5b}$$

In obtaining (3.5) from (3.3) we made use of the following representation[9] for $G_\mu(|\vec{r}-\vec{r}'|)$ valid for $\mathrm{Re}\mu>0$

and $\operatorname{Im}\mu > 0$:

$$G_\mu(|\vec{r}-\vec{r}'|) = \frac{i}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{w_\mu}\exp\{i[p(x-x')+q(y-y')+w_\mu|z-z'|]\}\,dp\,dq\,. \tag{3.6}$$

We now operate by $(\partial^2/\partial z^2 + w_\mu{}^2)$ on both sides of Eq. (3.5) to get

$$\left(\frac{\partial^2}{\partial z^2}+w_\mu{}^2\right)\left\{k_0{}^2\epsilon_0(\omega)\vec{E}(p,q;z,\omega)-\left(ip,iq,\frac{\partial}{\partial z}\right)\times\left[\left(ip,iq,\frac{\partial}{\partial z}\right)\times\vec{E}(p,q;z,\omega)\right]\right\}-\chi k_0{}^2\vec{E}(p,q;z,\omega)$$
$$= -\frac{eik_0}{\pi c}\,\hat{z}\left(w_\mu{}^2-\frac{\omega^2}{v^2}\right)e^{i(\omega/v)z}\,, \tag{3.7}$$

where we have used the fact that

$$(\partial^2/\partial z^2 + w_\mu{}^2)\exp(iw_\mu|z-z'|)/w_\mu = 2i\delta(z-z')\,. \tag{3.8}$$

Written in Cartesian components Eq. (3.7) is a set of three linear, ordinary, inhomogeneous, coupled differential equations with constant coefficients in the Cartesian components of the vector $\vec{E}(p,q;z,\omega)$. The general solution to Eq. (3.7) can therefore be written as

$$\vec{E}(p,q;z,\omega) = \vec{E}^{(H)}(p,q;z,\omega) + \vec{E}^{(P)}(p,q;z,\omega)\,, \tag{3.9}$$

where $\vec{E}^{(H)}(p,q;z,\omega)$ is the general solution to the homogeneous equation

$$\left(\frac{\partial^2}{\partial z^2}+w_\mu{}^2\right)\left\{k_0{}^2\epsilon_0(\omega)\vec{E}^{(H)}(p,q;z,\omega)-\left(ip,iq,\frac{\partial}{\partial z}\right)\times\left[\left(ip,iq,\frac{\partial}{\partial z}\right)\times\vec{E}^{(H)}(p,q;z,\omega)\right]\right\}$$
$$-\frac{\chi w^2}{c^2}\vec{E}^{(H)}(p,q;z,\omega) = 0\,, \tag{3.10}$$

and $\vec{E}^{(P)}(p,q;z,\omega)$ is any particular solution to (3.7), i.e.,

$$\left(\frac{\partial^2}{\partial z^2}+w_\mu{}^2\right)\left\{k_0{}^2\epsilon_0(\omega)\vec{E}^{(P)}(p,q;z,\omega)-\left(ip,iq,\frac{\partial}{\partial z}\right)\times\left[\left(ip,iq,\frac{\partial}{\partial z}\right)\times\vec{E}^{(P)}(p,q;z,\omega)\right]\right\}$$
$$-\frac{\chi\omega^2}{c^2}\vec{E}^{(P)}(p,q;z,\omega) = -\frac{eik_0}{\pi c}\,\hat{z}\left(w_\mu{}^2-\frac{\omega^2}{v^2}\right)e^{i(\omega/v)z}\,. \tag{3.11}$$

A lengthy but straightforward calculation shows that the following field is a particular solution to Eq. (3.11):

$$\vec{E}^{(P)}(p,q;z,\omega) = \vec{E}^{(P)}(p,q,\omega)e^{i(\omega/v)z}\,, \tag{3.12}$$

where

$$\vec{E}^{(P)}(p,q,\omega) = \frac{ei}{2\pi^2 v}\,\frac{\left(p,q,\dfrac{v}{\omega}\left(\dfrac{\omega^2}{v^2}-\dfrac{\omega^2}{c^2}\epsilon(\vec{k}_v,\omega)\right)\right)}{\epsilon(\vec{k}_v,\omega)[\vec{k}_v{}^2-k_0{}^2\epsilon(\vec{k}_v,\omega)]} \tag{3.13}$$

and

$$\vec{k}_v = (p,q,\omega/v)\,, \tag{3.14}$$

$$\epsilon(\vec{k}_v,\omega) \equiv \epsilon_0 + \frac{\chi}{\vec{k}_v{}^2-\mu^2(\omega)}\,. \tag{3.15}$$

The general solution to the Eq. (3.10) has been recently obtained in Ref. 8d as

$$\vec{E}^{(H)}(p,q,z,\omega) = \sum_{j=1}^{4}\vec{A}_j(p,q;\omega)e^{iw_jz} + \sum_{j=1}^{2}\vec{A}_j'(p,q;\omega)e^{iw_j'z}\,, \tag{3.16}$$

where $w_j$ and $w_j'$ are roots of the equations

$$\chi k_0{}^2 - (w_\mu{}^2-w_j{}^2)(w_\mu{}^2-w_j{}^2+k_0{}^2\epsilon_0) = 0\,, \tag{3.17}$$

and

$$w_\mu{}^2 - w_j'{}^2 - \frac{\chi}{\epsilon_0} = 0\,. \tag{3.18}$$

Moreover, the vector amplitudes $\vec{A}_j$ and $\vec{A}_j'$ satisfy the following conditions:

$$(p,q,w_j)\cdot\vec{A}_j(p,q;\omega) = 0\,, \tag{3.19}$$

and

$$(p,q,w_j')\times\vec{A}_j'(p,q;\omega) = 0\,. \tag{3.20}$$

Hence, Eq. (3.19) expresses the transversal character of the vector amplitudes $\vec{A}_j(p,q;\omega)$ with respect to the wave vector $(p, q, \omega_j)$ and Eq. (3.20) expresses the longitudinal character of the amplitude vectors $\vec{A}_j'(p, q; \omega)$ with respect to the wave vector $(p, q, w_j')$.

For the case when $\epsilon_0(\omega)$, $\chi$, $m_e^*$, $\Gamma$, $\omega$ are real and positive, it has been shown in Ref. 8d that of the four possible values of $w_j$, that satisfy Eq. (3.17), two have positive real and positive imaginary parts and the other two have negative real and negative imaginary parts. Of the two solutions $w_j'$ that satisfy Eq. (3.18), one has positive real and positive imaginary part and the other has negative real and negative imaginary part. This implies that the homogeneous field $\vec{E}^{(H)}(p, q; z, \omega)$ consist of six inhomogeneous plane waves, three of which propagate from the plane $z=0$ to the plane $z=d$, and the other three propagate from the planes $z=d$ towards the plane $z=0$. All of these inhomogeneous plane waves decay in the direction of their propagation.

Equation (3.9) together with Eqs. (3.12) and (3.16) represent the general solution to the equation (3.7) which by construction contains the general solution to the inhomogeneous integrodifferential equation (3.5). We therefore substitute the solutions of (3.7), namely, (3.9) into (3.5) and carry out the explicit integrations involved and use the Eqs. (3.17)–(3.20) to obtain the following equation:

$$\left[\sum_{j=1}^{4}\frac{\vec{A}_j}{w_\mu - w_j} + \sum_{j=1}^{2}\frac{\vec{A}_j'}{w_\mu - w_j'} + \frac{\vec{E}^{(P)}(p, q; \omega)}{w_\mu - \omega/v}\right]e^{iw_\mu z} = \left[\sum_{j=1}^{4}\frac{\vec{A}_j e^{iw_j d}}{w_\mu + w_j} + \sum_{j=1}^{2}\frac{\vec{A}_j' e^{iw_j' d}}{w_\mu + w_j'} + \frac{\vec{E}^{(P)}(p, q; \omega)e^{i(\omega/v)d}}{w_\mu + \omega/v}\right]e^{iw_\mu(d-z)}. \tag{3.21}$$

In order that (3.21) be satisfied for all values of $z$ in the range $0 \le z \le d$, the coefficients of $e^{iw_\mu z}$ and $e^{-iw_\mu z}$ must separately vanish, i.e.,

$$\sum_{j=1}^{4}\frac{\vec{A}_j}{w_\mu - w_j} + \sum_{j=1}^{2}\frac{\vec{A}_j'}{w_\mu - w_j'} + \frac{\vec{E}^{(P)}(p, q; \omega)}{w_\mu - \omega/v} = 0, \tag{3.22}$$

and

$$\sum_{j=1}^{4}\frac{\vec{A}_j \exp[i(w_j + w_\mu)d]}{w_\mu + w_j} + \sum_{j=1}^{2}\frac{\vec{A}_j' \exp[i(w_j' + w_\mu)d]}{w_\mu + w_j'} + \frac{\vec{E}^{(P)}(p, q; \omega)\exp[i(w_\mu + \omega/v)d]}{w_\mu + \omega/v} = 0. \tag{3.23}$$

Following the terminology introduced by Agarwal *et al.*[8d] we call (3.22) and (3.23) the *inhomogeneous-mode coupling conditions*. The solutions, as given by Eqs. (3.9), (3.12), (3.16)–(3.20), will be general solutions to the inhomogeneous integrodifferential equation (3.5) subject to the restrictions imposed by (3.22) and (3.23). The solutions to Eq. (3.3) can therefore be obtained if we use Eqs. (3.4), (3.9), (3.12), (3.16)–(3.20), (3.22)–(3.23), and one finds that

$$\vec{E}(\vec{r}, \omega) = \vec{E}_T^{(H)}(\vec{r}, \omega) + \vec{E}_L^{(H)}(\vec{r}, \omega) + \vec{E}^{(P)}(\vec{r}, \omega), \tag{3.24}$$

where

$$\vec{E}_T^{(H)}(\vec{r}, \omega) = \sum_{j=1}^{4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dp\,dq\,\vec{A}_j(p, q; \omega)e^{i\vec{k}_j\cdot\vec{r}}, \tag{3.25}$$

$$\vec{E}_L^{(H)}(\vec{r}, \omega) = \sum_{j=1}^{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dp\,dq\,\vec{A}_j'(p, q; \omega)e^{i\vec{k}_j'\cdot\vec{r}}, \tag{3.26}$$

$$\vec{E}^{(P)}(\vec{r}, \omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dp\,dq\,\vec{E}^{(P)}(p, q; \omega)e^{i\vec{k}_v\cdot\vec{r}}, \tag{3.27}$$

and

$$\vec{k}_j = (p, q, w_j), \tag{3.28}$$

$$\vec{k}_j' = (p, q, w_j'), \tag{3.29}$$

$$\vec{k}_v = (p, q, \omega/v), \tag{3.30}$$

with $w_j$ and $w_j'$ being the roots of Eqs. (3.17) and (3.18). $\vec{E}^{(P)}(p, q, \omega)$ is given by Eq. (3.13). The amplitudes $\vec{A}_j$ and $\vec{A}_j'$ obey the transversality condition (3.19) and the longitudinality condition (3.20), i.e.,

$$\vec{k}_j \cdot \vec{A}_j = 0. \tag{3.31}$$

and

$$\vec{k}_j' \times \vec{A}_j' = 0. \tag{3.32}$$

Besides the amplitudes $\vec{A}_j$ and $\vec{A}_j'$ are related by the inhomogeneous-mode coupling conditions (3.22) and (3.23).

Using (3.28)–(3.29) and (3.17)–(3.18) it can be easily seen that

$$(\vec{k}_j)^2 - k_0^2\epsilon(\vec{k}_j, \omega) = 0 \tag{3.33}$$

and

$$\epsilon(\vec{k}_j', \omega) = 0. \tag{3.34}$$

Equation (3.33) is the *transverse dispersion relation*, providing the permissible wave vectors $\vec{k}_j$ of the transverse waves and (3.34) is the *longitudinal dispersion relation*, satisfied by the wave vectors $\vec{k}_j'$ of the longitudinal waves. From (3.25)–(3.26) and (3.31) and (3.32) it can be verified that

$$\nabla \cdot \vec{E}_T^{(H)}(\vec{r}, \omega) = 0\,, \tag{3.35}$$

and

$$\nabla \times \vec{E}_L^{(H)}(\vec{r}, \omega) = 0\,. \tag{3.36}$$

We may summarize the results of this section by saying that the general mode structure in $(p, q, \omega)$ space for the electric field generated by a uniformly moving charged particle traversing perpendicular to this slab consists of three parts.

(1) A transverse homogeneous part which consists in general of four transverse inhomogeneous plane waves, two of which propagate from the plane $z=0$ to the plane $z=d$ and the other two from the plane $z=d$ to the plane $z=0$.

(2) A longitudinal homogeneous part which consists of two longitudinal plane waves, one of which propagate from the plane $z=0$ to the plane $z=d$, and the other from the plane $z=d$ to the plane $z=0$.

(3) A particular solution which consists of a plane wave traveling in the positive $z$ direction with the speed $v$ of the charged particle.

The amplitudes of the inhomogeneous plane waves are not arbitrary, but are coupled by the two *inhomogeneous-mode coupling conditions* (3.22)–(3.23). Moreover, in general each of these inhomogeneous plane waves decay in the direction of their propagation.

## IV. ČERENKOV AND TRANSITION RADIATION FIELDS

In Sec. III, we discussed the nature of the field associated with a uniformly moving charged particle passing perpendicularly through our spatially dispersive slab. The electromagnetic field inside the slab was found in terms of angular spectrum representations, and the amplitudes of the waves were found to obey the inhomogeneous-mode coupling conditions (3.22)–(3.23), the transversality condition (3.31), and the longitudinality condition (3.32). All these conditions do not completely determine all the spectral amplitudes involved. The rest of the spectral amplitudes will be specified by the Maxwell continuity conditions at the two faces of the slab. However, to simplify the mathematics we will choose the geometry of a half-space, i.e., we will take the limit $d \to \infty$ in the results of Sec. III. This in physical terms will imply that the amplitude of waves traveling from the face at $z=\infty$ towards the face at $z=0$ be put equal to zero. After obtaining the proper mode expansion of the electromagnetic field inside the spatially dispersive half-space, we write down the corresponding mode expansion in the opposite half-space which is taken for simplicity to be vacuum. We then apply the continuity conditions of Maxwell theory and determine all the as yet undetermined spectral amplitudes and identify both the Čerenkov and the transition radiation fields.

The general mode structure for electromagnetic fields assoiciated with a uniformly moving charged particle in a vacuum is known.[10] For the domain $-\infty < z < 0$, $-\infty < x < +\infty$, $-\infty < y < +\infty$, the electric field associated with the charged particle moving with velocity $\vec{v} = v\hat{z}$ and having a charge density $\rho(\vec{r}, t) = \delta(x)\delta(y)\delta(z - vt)$ can be represented as

$$\vec{E}^{(-)}(\vec{r}, \omega) = \vec{E}_{TR}^{(-)}(\vec{r}, \omega) + \vec{E}_c^{(-)}(\vec{r}, \omega)\,, \tag{4.1}$$

where

$$\vec{E}_{TR}^{(-)}(\vec{r}, \omega) = \int_{-\infty}^{+} \int_{-\infty}^{+\infty} \vec{E}_{TR}^{(-)}(p, q; \omega) e^{i\vec{k}^{(-)}\cdot\vec{r}} dp\,dq\,, \tag{4.2}$$

$$\vec{E}_c^{(-)}(\vec{r}, \omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \vec{E}_c^{(-)}(p, q; \omega) e^{i\vec{k}_v\cdot\vec{r}} dp\,dq\,, \tag{4.3}$$

$$\vec{k}^{(-)} = (p, q, -w_0)\,,$$

where

$$w_0 = (k_0^2 - p^2 - q^2)^{1/2} \text{ if } p^2 + q^2 < k_0^2$$

$$= i(p^2 + q^2 - k_0^2)^{1/2} \text{ if } p^2 + q^2 > k_0^2\,, \tag{4.4}$$

and

$$\vec{k}_v = (p, q, \omega/v)\,. \tag{4.5}$$

The representations (4.1)–(4.5) can be recognized as the angular spectrum representations[10] which are found useful in many problems involving fields in a half-space. The angular spectrum amplitudes $\vec{E}_{TR}^{(-)}(p, q; \omega)$ have yet to be determined, whereas for the particular solution $\vec{E}_c^{(-)}(p, q; \omega)$, we take[11]

$$\vec{E}_c^{(-)}(p, q, \omega) = \frac{ei}{2\pi^2 v} \frac{\left(p, q, \frac{\omega}{v}\left(1 - \frac{v^2}{c^2}\right)\right)}{\vec{k}_v^{\,2} - k_0^2}. \tag{4.6}$$

Using (4.1), (4.6), and the vacuum Maxwell equation

$$\nabla \cdot \vec{E}^{(-)}(\vec{r}, t) = 4\pi\rho(\vec{r}, t)\,, \tag{4.7}$$

with

$$\rho(\vec{r}, t) = \delta(x)\delta(y)\delta(z - vt) \tag{4.8}$$

appropriate to our case, it can be shown that

$$\vec{k}^{(-)} \cdot \vec{E}_{TR}^{(-)}(p, q; \omega) = 0\,. \tag{4.9}$$

The magnetic field vector $\vec{H}^{(-)}(\vec{r}, \omega)$ can be easily determined from the Maxwell equation

$$\vec{H}^{(-)}(\vec{r}, \omega) = \frac{i}{k_0} \nabla \times \vec{E}^{(-)}(\vec{r}, \omega)\,. \tag{4.10}$$

In order to obtain a representation for the fields inside the model spatially dispersive medium which occupies the half-space $z>0$, we discard the generalized plane waves in (3.25) and (3.26) with $\vec{k}$ vectors whose $z$ components have negative real and imaginary parts, as these are obviously waves that are reflected from the rear surface of the slab and must be absent in the limiting case when the thickness $d \to \infty$, i.e., when the slab degenerates into a half-space. So we have the following mode expansion associated with the moving charged particle in the half-space:

$$\vec{E}^{(+)}(\vec{r}, \omega) = \vec{E}_{TR}^{(+)}(\vec{r}, \omega) + \vec{E}_{P}^{(+)}(\vec{r}, \omega)\,, \tag{4.11}$$

where

$$\vec{E}^{(+)}(\vec{r}, \omega) = \sum_{j=1}^{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dp\, dq\, \vec{A}_j(p, q; \omega) e^{i\vec{k}_j^{(+)} \cdot \vec{r}} + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dp\, dq\, \vec{A}_l(p, q; \omega) e^{i\vec{k}_l^{(+)} \cdot \vec{r}}\,, \tag{4.12}$$

$$\vec{k}_j^{(H)} = (p, q, w_j)\,, \quad j = 1, 2\,. \tag{4.13}$$

$$\vec{k}_l^{(+)} = (p, q, w_l)\,, \tag{4.14}$$

and

$$\vec{E}_P^{(+)}(\vec{r}, \omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dp\, dq\, \vec{E}_P^{(+)}(p, q; \omega) e^{i\vec{k}_v \cdot \vec{r}}\,, \tag{4.15a}$$

$$\vec{E}_P^{(+)}(p, q; \omega) = \frac{ei}{2\pi^2 v} \frac{\left(p, q, \frac{v}{\omega}\left(\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\epsilon(\vec{k}_v, \omega)\right)\right)}{\epsilon(\vec{k}_v, \omega)\left(\vec{k}_v^{\,2} - \frac{\omega^2}{c^2}\epsilon(\vec{k}_v, \omega)\right)}\,. \tag{4.15b}$$

Here $w_j$ are those roots of (3.33) with $\mathrm{Re}\,w_j > 0$, $\mathrm{Im}\,w_j > 0$ (there are two such roots), and $w_l$ is the root of Eq. (3.34) with $\mathrm{Re}\,w_l > 0$, $\mathrm{Im}\,w_l > 0$ (there being one such root).

As we already pointed out, the amplitudes $\vec{A}_j$ and $\vec{A}_l$ are not quite arbitrary. $\vec{A}_j$ satisfies the transversality condition (3.31), i.e.,

$$\vec{k}_j^{(+)} \cdot \vec{A}_j = 0\,, \tag{4.16}$$

and $\vec{A}_l$ satisfies the longitudinality condition (3.32), i.e.,

$$\vec{k}_l^{(+)} \times \vec{A}_l = 0\,. \tag{4.17}$$

Moreover, $\vec{A}_j$ and $\vec{A}_l$ are related by the mode coupling conditions (3.22) and (3.33) which in the limiting case $d \to \infty$ (for the half-space) reduce to the single condition

$$\sum_{j=1}^{2} \frac{\vec{A}_j}{w_j - w_\mu} + \frac{\vec{A}_l}{w_l - w_\mu} + \frac{\vec{E}_P^{(+)}}{\omega/v - w_\mu} = 0\,. \tag{4.18}$$

[The Eq. (3.23) in this limiting case becomes a trivial identity.]

The magnetic field inside the spatially dispersive half-space can be obtained from the equation

$$\vec{H}^{(+)}(\vec{r}, \omega) = \frac{i}{k_0} \nabla \times \vec{E}^{(+)}(\vec{r}, \omega)\,. \tag{4.19}$$

Now in order to obtain expressions for the arbitrary constants which enter in (4.2), (4.12), we make use of (1) continuity conditions of Maxwell's theory at the interface $z=0$, i.e.,

$$(\hat{z} \times \vec{E}^{(-)} = \hat{z} \times \vec{E}^{(+)})_{z=0}\,, \tag{4.20}$$

$$(\hat{z} \times \vec{H}^{(-)} = \hat{z} \times \vec{H}^{(+)})_{z=0}\,, \tag{4.21}$$

where the plus and minus signs denote limits taken from $+z$ or $-z$, (2) the transversality conditions (4.9), (4.16), (3) the longitudinality condition (4.17), and (4) the mode coupling conditions (4.18). We then find after a straightforward, but lengthy calculation the following results:

$$[\vec{E}_{TR}^{(-)}(p, q, \omega)]_x = \frac{1}{\Delta} \begin{vmatrix} \mathcal{E}_x & 0 & -\alpha(1-\beta p^2) & \alpha\beta pq \\ \frac{q}{p}\mathcal{E}_x & 1 & \alpha\beta pq & -\alpha(1-\beta q^2) \\ \mathcal{E}_x' & \frac{pq}{w_0} & Q + p^2\delta & pq\delta \\ \frac{q}{p}\mathcal{E}_x' & \frac{q^2 + w_0^2}{w_0} & pq\delta & Q + q^2\delta \end{vmatrix}\,, \tag{4.22}$$

$$[\vec{E}_{TR}^{(-)}(p,q,\omega)]_y = \frac{1}{\Delta}\begin{vmatrix} 1 & \mathcal{E}_x & -\alpha(1-\beta p^2) & \alpha\beta pq \\ 0 & \frac{q}{p}\mathcal{E}_x & \alpha\beta pq & -\alpha(1-\beta q^2) \\ \frac{p^2+w_0{}^2}{w_0} & \mathcal{E}'_x & Q+p^2\delta & pq\delta \\ \frac{pq}{w_0} & \frac{q}{p}\mathcal{E}'_x & pq\delta & Q+q^2\delta \end{vmatrix}, \tag{4.23}$$

$$[\vec{E}_{TR}^{(-)}(p,q,\omega)]_z = -\frac{p}{m_0}[\vec{E}_{TR}^{(-)}(p,q,\omega)]_x - \frac{q}{m_0}[\vec{E}_{TR}^{(-)}(p,q,\omega)]_y, \tag{4.24}$$

$$\vec{A}_1 = \frac{w_\mu - w_1}{w_\mu - w_2}\left[\frac{\vec{k}_l^{(+)}(\vec{k}_1^{(+)}\cdot\vec{A}_2)}{\vec{k}_l^{(+)}\cdot\vec{k}_1^{(+)}} - \vec{A}_2\right], \tag{4.25}$$

$$\vec{A}_2 = \left(A_{2x}, A_{2y}, -\frac{pA_{2x}+qA_{2y}}{w_2}\right) \tag{4.26}$$

$$\vec{A}_l = \vec{k}_l^{(+)}\left[\frac{w_l - w_\mu}{w_\mu - w_2}\frac{\vec{k}_1^{(+)}\cdot\vec{A}_2}{\vec{k}_l^{(+)}\cdot\vec{k}_1^{(+)}} + \frac{w_l - w_\mu}{w_\mu - \omega/v}\frac{\vec{k}_1^{(+)}\cdot\vec{E}_p^{(+)}}{\vec{k}_1^{(+)}\cdot\vec{k}_l^{(+)}}\right], \tag{4.27}$$

where

$$A_{2x} = \frac{1}{\Delta}\begin{vmatrix} 1 & 0 & \mathcal{E}_x & \alpha\beta pq \\ 0 & 1 & \frac{q}{p}\mathcal{E}_x & -\alpha(1-\beta q^2) \\ \frac{p^2+w_0{}^2}{w_0} & \frac{pq}{w_0} & \mathcal{E}'_x & pq\delta \\ \frac{pq}{w_0} & \frac{q^2+w_0{}^2}{w_0} & \frac{q}{p}\mathcal{E}'_x & Q+q^2\delta \end{vmatrix}, \tag{4.28}$$

$$A_{2y} = \frac{1}{\Delta}\begin{vmatrix} 1 & 0 & -\alpha(1-\beta p^2) & \mathcal{E}_x \\ 0 & 1 & \alpha\beta pq & \frac{q}{p}\mathcal{E}_x \\ \frac{p^2+w_0{}^2}{w_0} & \frac{pq}{w_0} & Q+p^2\delta & \mathcal{E}'_x \\ \frac{pq}{w_0} & \frac{q^2+w_0{}^2}{w_0} & pq\delta & \frac{q}{p}\mathcal{E}'_x \end{vmatrix}, \tag{4.29}$$

$$\alpha = \frac{w_1 - w_2}{w_\mu - w_2}, \quad \beta = \frac{w_l - w_1}{(\vec{k}_1^{(+)}\cdot\vec{k}_l^{(+)})w_2}, \quad \gamma = w_\mu - w_1, \quad \delta = \alpha\beta\gamma + \frac{\alpha}{w_2}; \quad Q = w_1(1+\alpha) - w_2, \tag{4.30}$$

$$\Delta = Q^2 + Q\left[\delta(p^2+q^2) - \frac{\alpha\beta}{w_0}(p^2+q^2)^2 + \frac{\alpha}{w_0}(p^2+q^2) + 2\alpha w_0 - \alpha\beta w_0(p^2+q^2)\right]$$
$$+ \alpha\delta(p^2+q^2)w_0 + \alpha^2 k_0{}^2[1-\beta(p^2+q^2)], \tag{4.31}$$

$$\mathcal{E}_x = \frac{w_1 - \omega/v}{w_\mu - \omega/v}(\vec{E}_p^{(+)})_x + \frac{p(w_l - w_1)}{w_\mu - \omega/v}\frac{\vec{k}_1^{(+)}\cdot\vec{E}_p^{(+)}}{\vec{k}_1^{(+)}\cdot\vec{k}_l^{(+)}} - (\vec{E}_c^{(-)})_x, \tag{4.32}$$

and

$$\mathcal{E}'_x = \frac{w_\mu - w_1}{w_\mu - \omega/v}\left[p\frac{w_l - w_1}{w_\mu - \omega/v}\frac{\vec{k}_1^{(+)}\cdot\vec{E}_p^{(+)}}{\vec{k}_1^{(+)}\cdot\vec{k}_l^{(+)}} - p(\vec{E}_p^{(+)})_z - w_1(\vec{E}_p^{(+)})_x\right]$$
$$+ p[(\vec{E}_c^{(-)})_z - (\vec{E}_p^{(+)})_z] + \frac{\omega}{v}[(\vec{E}_p^{(+)})_x - (\vec{E}_c^{(-)})_x]. \tag{4.33}$$

Equations (4.22)–(4.33) give expressions for the angular spectrum amplitudes of the homogeneous part of the fields outside and inside the spatially dispersive medium, generated by the passage of the uniformly moving electron. The homogeneous field $\vec{E}_{TR}^{(+)}$ inside the medium is a superposition of plane-wave fields that propagate with velocities $c/n_1$, $c/n_2$, and $c/n_l$. The refractive indices $n_1$, $n_2$, and $n_l$ are related to the wave numbers $k_1^{(+)}$, $k_2^{(+)}$, and $k_l^{(+)}$ in the usual way; for example,

$$n_1 = \frac{k_1^{(+)}}{k_0} \quad \text{etc.}$$

The homogeneous fields $\vec{E}_{TR}^{(-)}$ and $\vec{E}_{TR}^{(+)}$ represent *transition radiation*[2] being produced by the transition of the moving electron from the vacuum into the medium. The particular field $\vec{E}_p^{(+)}$ inside the medium which travels with the velocity, $v$, of the charged particle represents the *Čerenkov field.* In paper II of this investigation we will obtain closed-form expressions for the particular field and threshold conditions for the Čerenkov radiation inside the medium. We will also obtain asymptotic expansions of the transition radiation field and discuss other properties associated with these fields.

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